Elementary Differential Equations and Boundary Value Problems

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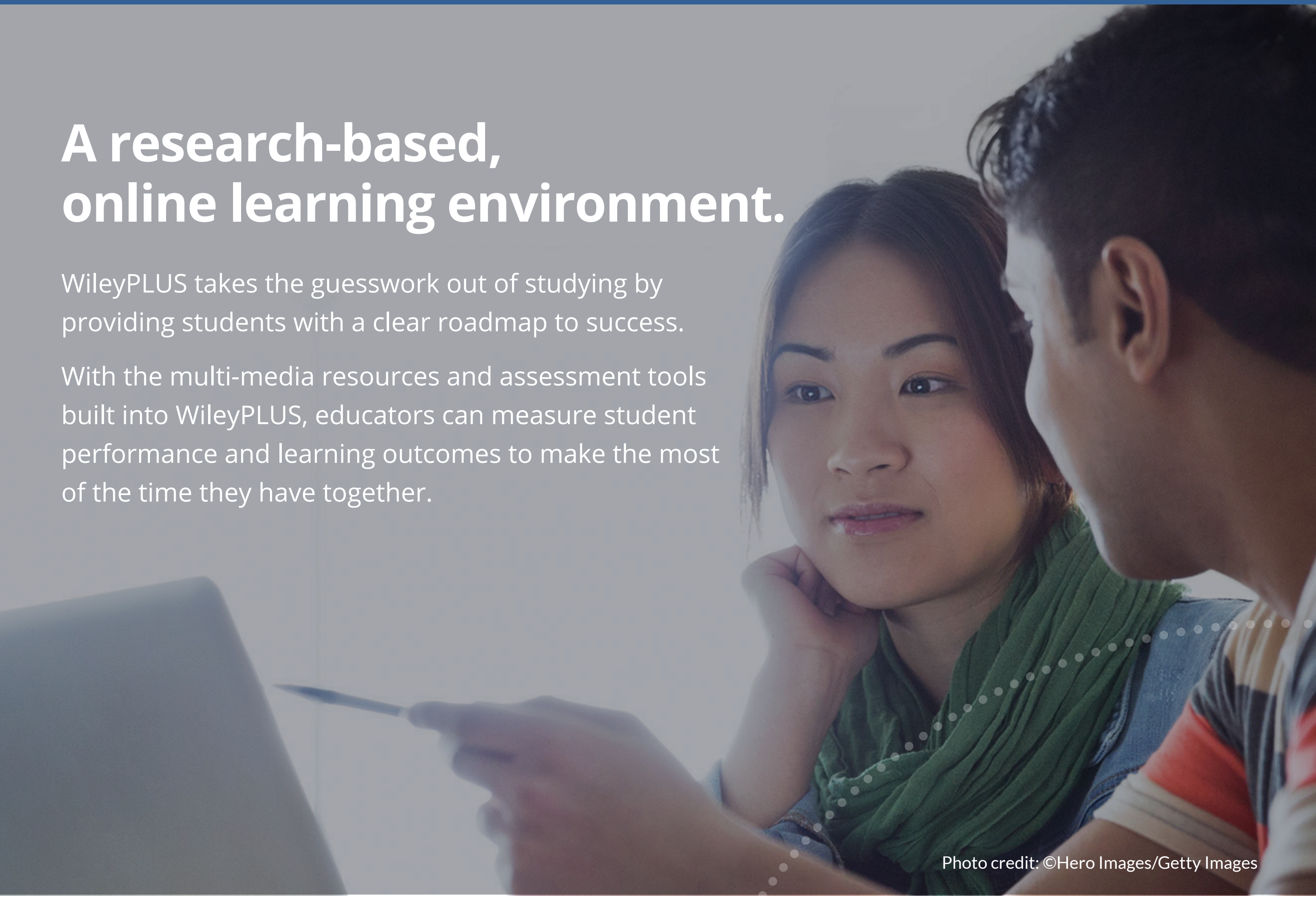

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Elementary Differential Equations and Boundary Value Problems

Eleventh Edition

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To Elsa, Betsy, and in loving memory of Maureen

To Siobhan, James, Richard Jr., Carolyn, Ann, Stuart,
Michael, Marybeth, and Bradley

And to the next generation:
Charles, Aidan, Stephanie, Veronica, and Deirdre

The Authors

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RICHARD C. DIPRIMA (deceased) received his B.S., M.S., and Ph.D. degrees in Mathematics from Carnegie-Mellon University. He joined the faculty of Rensselaer Polytechnic Institute after holding research positions at MIT, Harvard, and Hughes Aircraft. He held the Eliza Ricketts Foundation Professorship of Mathematics at Rensselaer, was a fellow of the American Society of Mechanical Engineers, the American Academy of Mechanics, and the American Physical Society. He was also a member of the American Mathematical Society, the Mathematical Association of America, and the Society for Industrial and Applied Mathematics. He served as the Chairman of the Department of Mathematical Sciences at Rensselaer, as President of the Society for Industrial and Applied Mathematics, and as Chairman of the Executive Committee of the Applied Mechanics Division of ASME. In 1980, he was the recipient of the William H. Wiley Distinguished Faculty Award given by Rensselaer. He received Fulbright fellowships in 1964–65 and 1983 and a Guggenheim fellowship in 1982–83. He was the author of numerous technical papers in hydrodynamic stability and lubrication theory and two texts on differential equations and boundary value problems. Professor DiPrima died on September 10, 1984.

DOUGLAS B. MEADE received B.S. degrees in Mathematics and Computer Science from Bowling Green State University, an M.S. in Applied Mathematics from Carnegie Mellon University, and a Ph.D. in mathematics from Carnegie Mellon University. After a two-year stint at Purdue University, he joined the mathematics faculty at the University of South Carolina, where he is currently an Associate Professor of mathematics and the Associate Dean for Instruction, Curriculum, and Assessment in the College of Arts and Sciences. He is a member of the American Mathematical Society, Mathematics Association of America, and Society for Industrial and Applied Mathematics; in 2016 he was named an ICTCM Fellow at the International Conference on Technology in Collegiate Mathematics (ICTCM). His primary research interests are in the numerical solution of partial differential equations arising from wave propagation problems in unbounded domains and from population models for infectious diseases. He is also well-known for his educational uses of computer algebra systems, particularly Maple. These include Getting Started with Maple (with M. May, C-K. Cheung, and G. E. Keough, Wiley, 2009, ISBN 978-0-470-45554-8), Engineer’s Toolkit: Maple for Engineers (with E. Bourkoff, Addison-Wesley, 1998, ISBN 0-8053-6445-5), and numerous Maple supplements for numerous calculus, linear algebra, and differential equations textbooks - including previous editions of this book. He was a member of the MathDL New Collections Working Group for Single Variable Calculus, and chaired the Working Groups for Differential Equations and Linear Algebra. The NSF is partially supporting his work, together with Prof. Philip Yasskin (Texas A&M), on the Maplets for Calculus project.

Preface

As we have prepared an updated edition our first priorities are to preserve, and to enhance, the qualities that have made previous editions so successful. In particular, we adopt the viewpoint of an applied mathematician with diverse interests in differential equations, ranging from quite theoretical to intensely practical – and usually a combination of both. Three pillars of our presentation of the material are methods of solution, analysis of solutions, and approximations of solutions. Regardless of the specific viewpoint adopted, we have sought to ensure the exposition is simultaneously correct and complete, but not needlessly abstract.

The intended audience is undergraduate STEM students whose degree program includes an introductory course in differential equations during the first two years. The essential prerequisite is a working knowledge of calculus, typically a two- or three-semester course sequence or an equivalent. While a basic familiarity with matrices is helpful, Sections 7.2 and 7.3 provide an overview of the essential linear algebra ideas needed for the parts of the book that deal with systems of differential equations (the remainder of Chapter 7, Section 8.5, and Chapter 9).

A strength of this book is its appropriateness in a wide variety of instructional settings. In particular, it allows instructors flexibility in the selection of and the ordering of topics and in the use of technology. The essential core material is Chapter 1, Sections 2.1 through 2.5, and Sections 3.1 through 3.5. After completing these sections, the selection of additional topics, and the order and depth of coverage are generally at the discretion of the instructor. Chapters 4 through 11 are essentially independent of each other, except that Chapter 7 should precede Chapter 9, and Chapter 10 should precede Chapter 11.

A particularly appealing aspect of differential equations is that even the simplest differential equations have a direct correspondence to realistic physical phenomena: exponential growth and decay, spring-mass systems, electrical circuits, competitive species, and wave propagation. More complex natural processes can often be understood by combining and building upon simpler and more basic models. A thorough knowledge of these basic models, the differential equations that describe them, and their solutions – either explicit solutions or qualitative properties of the solution – is the first and indispensable step toward analyzing the solutions of more complex and realistic problems. The modeling process is detailed in Chapter 1 and Section 2.3. Careful constructions of models appear also in Sections 2.5, 3.7, 9.4, 10.5, and 10.7 (and the appendices to Chapter 10). Various problem sets throughout the book include problems that involve modeling to formulate an appropriate differential equation, and then to solve it or to determine some qualitative properties of its solution. The primary purposes of these applied problems are to provide students with hands-on experience in the derivation of differential equations, and to convince them that differential equations arise naturally in a wide variety of real-world applications.

Another important concept emphasized repeatedly throughout the book is the transportability of mathematical knowledge. While a specific solution method applies to only a particular class of differential equations, it can be used in any application in which that particular type of differential equation arises. Once this point is made in a convincing manner, we believe that it is unnecessary to provide specific applications of every method of solution or type of equation that we consider. This decision helps to keep this book to a reasonable size, and allows us to keep the primary emphasis on the development of more solution methods for additional types of differential equations.

From a student's point of view, the problems that are assigned as homework and that appear on examinations define the course. We believe that the most outstanding feature of this book is the number, and above all the variety and range, of the problems that it contains. Many problems are entirely straightforward, but many others are more challenging, and some are fairly open-ended and can even serve as the basis for independent student projects. The observant reader will notice that there are fewer problems in this edition than in previous editions; many of these problems remain available to instructors via the WileyPlus course. The remaining 1600 problems are still far more problems than any instructor can use in any given course, and this provides instructors with a multitude of choices in tailoring their course to meet their own goals and the needs of their students. The answers to almost all of these problems can be found in the pages at the back of the book; full solutions are in either the Student's Solution Manual or the Instructor's Solution Manual.

While we make numerous references to the use of technology, we do so without limiting instructor freedom to use as much, or as little, technology as they desire. Appropriate technologies include advanced graphing calculators (TI Nspire), a spreadsheet (Excel), web-based resources (applets), computer algebra systems, (Maple, Mathematica, Sage), scientific computation systems (MATLAB), or traditional programming (FORTRAN, Javascript, Python). Problems marked with a G are ones we believe are best approached with a graphical tool; those marked with a N are best solved with the use of a numerical tool. Instructors should consider setting their own policies, consistent with their interests and intents about student use of technology when completing assigned problems.

Many problems in this book are best solved through a combination of analytic, graphic, and numeric methods. Pencil-and-paper methods are used to develop a model that is best solved (or analyzed) using a symbolic or graphic tool. The quantitative results and graphs, frequently produced using computer-based resources, serve to illustrate and to clarify conclusions that might not be readily apparent from a complicated explicit solution formula. Conversely, the implementation of an efficient numerical method to obtain an approximate solution typically requires a good deal of preliminary analysis–to determine qualitative features of the solution as a guide to computation, to investigate limiting or special cases, or to discover ranges of the variables or parameters that require an appropriate combination of both analytic and numeric computation. Good judgment may well be required to determine the best choice of solution methods in each particular case. Within this context we point out that problems that request a "sketch" are generally intended to be completed without the use of any technology (except your writing device).

We believe that it is important for students to understand that (except perhaps in courses on differential equations) the goal of solving a differential equation is seldom simply to obtain the solution. Rather, we seek the solution in order to obtain insight into the behavior of the process that the equation purports to model. In other words, the solution is not an end in itself. Thus, we have included in the text a great many problems, as well as some examples, that call for conclusions to be drawn about the solution. Sometimes this takes the form of finding the value of the independent variable at which the solution has a certain property, or determining the long-term behavior of the solution. Other problems ask for the effect of variations in a parameter, or for the determination of all values of a parameter at which the solution experiences a substantial change. Such problems are typical of those that arise in the applications of differential equations, and, depending on the goals of the course, an instructor has the option of assigning as few or as many of these problems as desired.

Readers familiar with the preceding edition will observe that the general structure of the book is unchanged. The minor revisions that we have made in this edition are in many cases the result of suggestions from users of earlier editions. The goals are to improve the clarity and readability of our presentation of basic material about differential equations and their applications. More specifically, the most important revisions include the following:

1. Chapter 1 has been rewritten. Instead of a separate section on the History of Differential Equations, this material appears in three installments in the remaining three section.
2. Additional words of explanation and/or more explicit details in the steps in a derivation have been added throughout each chapter. These are too numerous and widespread to mention individually, but collectively they should help to make the book more readable for many students.
3. There are about forty new or revised problems scattered throughout the book. The total number of problems has been reduced by about 400 problems, which are still available through WileyPlus, leaving about 1600 problems in print.
4. There are new examples in Sections 2.1, 3.8, and 7.5.
5. The majority (is this correct?) of the figures have been redrawn, mainly by the use full color to allow for easier identification of critical properties of the solution. In addition, numerous captions have been expanded to clarify the purpose of the figure without requiring a search of the surrounding text.
6. There are several new references, and some others have been updated.

The authors have found differential equations to be a never-ending source of interesting, and sometimes surprising, results and phenomena. We hope that users of this book, both students and instructors, will share our enthusiasm for the subject.

William E. Boyce and Douglas B. Meade
Watervliet, New York and Columbia, SC
29 August 2016

Supplemental Resources for Instructors and Students

An Instructor's Solutions Manual, ISBN 978-1-119-16976-5, includes solutions for all problems not contained in the Student Solutions Manual.

A Student Solutions Manual, ISBN 978-1-119-16975-8, includes solutions for selected problems in the text.

A Book Companion Site, www.wiley.com/college/boyce, provides a wealth of resources for students and instructors, including

- PowerPoint slides of important definitions, examples, and theorems from the book, as well as graphics for presentation in lectures or for study and note taking.
- Chapter Review Sheets, which enable students to test their knowledge of key concepts. For further review, diagnostic feedback is provided that refers to pertinent sections in the text.
- Mathematica, Maple, and MATLAB data files for selected problems in the text providing opportunities for further exploration of important concepts.
- Projects that deal with extended problems normally not included among traditional topics in differential equations, many involving applications from a variety of disciplines. These vary in length and complexity, and they can be assigned as individual homework or as group assignments.

A series of supplemental guidebooks, also published by John Wiley & Sons, can be used with Boyce/DiPrima/Meade in order to incorporate computing technologies into the course. These books emphasize numerical methods and graphical analysis, showing how these methods enable us to interpret solutions of ordinary differential equations (ODEs) in the real world. Separate guidebooks cover each of the three major mathematical software formats, but the ODE subject matter is the same in each.

- Hunt, Lipsman, Osborn, and Rosenberg, *Differential Equations with MATLAB* , 3rd ed., 2012, ISBN 978-1-118-37680-5

- Hunt, Lardy, Lipsman, Osborn, and Rosenberg, *Differential Equations with Maple*, 3rd ed., 2008, ISBN 978-0-471-77317-7
- Hunt, Outing, Lipsman, Osborn, and Rosenberg, *Differential Equations with Mathematica*, 3rd ed., 2009, ISBN 978-0-471-77316-0

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- Homework management tools, which enable instructors easily to assign and grade questions, as well as to gauge student comprehension.
- QuickStart pre-designed reading and homework assign ments. Use them as is, or customize them to fit the needs of your classroom.

Acknowledgments

It is a pleasure to express my appreciation to the many people who have generously assisted in various ways in the preparation of this book.

To the individuals listed below, who reviewed the manuscript and/or provided valuable suggestions for its improvement:

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WILLIAM E. BOYCE AND DOUGLAS B. MEADE

Brief Contents

Contents

CHAPTER **1**

Introduction

In this first chapter we provide a foundation for your study of differential equations in several different ways. First, we use two problems to illustrate some of the basic ideas that we will return to, and elaborate upon, frequently throughout the remainder of the book. Later, to provide organizational structure for the book, we indicate several ways of classifying differential equations.

The study of differential equations has attracted the attention of many of the world's greatest mathematicians during the past three centuries. On the other hand, it is important to recognize that differential equations remains a dynamic field of inquiry today, with many interesting open questions. We outline some of the major trends in the historical development of the subject and mention a few of the outstanding mathematicians who have contributed to it. Additional biographical information about some of these contributors will be highlighted at appropriate times in later chapters.

1.1 Some Basic Mathematical Models; Direction Fields

Before embarking on a serious study of differential equations (for example, by reading this book or major portions of it), you should have some idea of the possible benefits to be gained by doing so. For some students the intrinsic interest of the subject itself is enough motivation, but for most it is the likelihood of important applications to other fields that makes the undertaking worthwhile.

Many of the principles, or laws, underlying the behavior of the natural world are statements or relations involving rates at which things happen. When expressed in mathematical terms, the relations are equations and the rates are derivatives. Equations containing derivatives are **differential equations**. Therefore, to understand and to investigate problems involving the motion of fluids, the flow of current in electric circuits, the dissipation of heat in solid objects, the propagation and detection of seismic waves, or the increase or decrease of populations, among many others, it is necessary to know something about differential equations.

A differential equation that describes some physical process is often called a **mathematical model** of the process, and many such models are discussed throughout this book. In this section we begin with two models leading to equations that are easy to solve. It is noteworthy that even the simplest differential equations provide useful models of important physical processes.

EXAMPLE 1 | A Falling Object

Suppose that an object is falling in the atmosphere near sea level. Formulate a differential equation that describes the motion.

Solution:

We begin by introducing letters to represent various quantities that may be of interest in this problem. The motion takes place during a certain time interval, so let us use t to denote time. Also, let us use v to represent the velocity of the falling object. The velocity will presumably change with time, so we think of v as a function of t; in other words, t is the independent variable and v is the dependent variable. The choice of units of measurement is somewhat arbitrary, and there is nothing in the statement of the problem to suggest appropriate units, so we are free to make any choice that seems reasonable. To be specific, let us measure time t in seconds and velocity v in meters/second. Further, we will assume that v is positive in the downward direction—that is, when the object is falling.

The physical law that governs the motion of objects is **Newton's second law**, which states that the mass of the object times its acceleration is equal to the net force on the object. In mathematical terms this law is expressed by the equation

$$F = ma, \tag{1}$$

where m is the mass of the object, a is its acceleration, and F is the net force exerted on the object. To keep our units consistent, we will measure m in kilograms, a in meters/second^2, and F in newtons. Of course, a is related to v by $a = dv/dt$, so we can rewrite equation (1) in the form

$$F = m\frac{dv}{dt}. \tag{2}$$

Next, consider the forces that act on the object as it falls. Gravity exerts a force equal to the weight of the object, or mg, where g is the acceleration due to gravity. In the units we have chosen, g has been determined experimentally to be approximately equal to 9.8 m/s^2 near the earth's surface.

There is also a force due to air resistance, or drag, that is more difficult to model. This is not the place for an extended discussion of the drag force; suffice it to say that it is often assumed that the drag is proportional to the velocity, and we will make that assumption here. Thus the drag force has the magnitude γv, where γ is a constant called the drag coefficient. The numerical value of the drag coefficient varies widely from one object to another; smooth streamlined objects have much smaller drag coefficients than rough blunt ones. The physical units for γ are mass/time, or kg/s for this problem; if these units seem peculiar, remember that γv must have the units of force, namely, kg·m/s^2.

In writing an expression for the net force F, we need to remember that gravity always acts in the downward (positive) direction, whereas, for a falling object, drag acts in the upward (negative) direction, as shown in Figure 1.1.1. Thus

$$F = mg - \gamma v \tag{3}$$

and equation (2) then becomes

$$m\frac{dv}{dt} = mg - \gamma v. \tag{4}$$

Differential equation (4) is a mathematical model for the velocity v of an object falling in the atmosphere near sea level. Note that the model contains the three constants m, g, and γ. The constants m and γ depend very much on the particular object that is falling, and they are usually different for different objects. It is common to refer to them as parameters, since they may take on a range of values during the course of an experiment. On the other hand, g is a physical constant, whose value is the same for all objects.

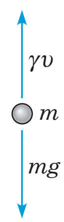

FIGURE 1.1.1 Free-body diagram of the forces on a falling object.

To solve equation (4), we need to find a function $v = v(t)$ that satisfies the equation. It is not hard to do this, and we will show you how in the next section. For the present, however, let us see what we can learn about solutions without actually finding any of them. Our task is simplified slightly if we assign numerical values to m and γ, but the procedure is the same regardless of which values we choose. So, let us suppose that $m = 10$ kg and $\gamma = 2$ kg/s. Then equation (4) can be rewritten as

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}. \tag{5}$$

EXAMPLE 2 | A Falling Object (continued)

Investigate the behavior of solutions of equation (5) without solving the differential equation.

Solution:

First let us consider what information can be obtained directly from the differential equation itself. Suppose that the velocity v has a certain given value. Then, by evaluating the right-hand side of differential equation (5), we can find the corresponding value of dv/dt. For instance, if $v = 40$, then $dv/dt = 1.8$. This means that the slope of a solution $v = v(t)$ has the value 1.8 at any point where $v = 40$. We can display this information graphically in the tv-plane by drawing short line segments with slope 1.8 at several points on the line $v = 40$. (See Figure 1.1.2(a)). Similarly, when $v = 50$, then $dv/dt = -0.2$, and when $v = 60$, then $dv/dt = -2.2$, so we draw line segments with slope -0.2 at several points on the line $v = 50$ (see Figure 1.1.2(b)) and line segments with slope -2.2 at several points on the line $v = 60$ (see Figure 1.1.2(c)). Proceeding in the same way with other values of v we create what is called a **direction field**, or a **slope field**. The direction field for differential equation (5) is shown in Figure 1.1.3.

Remember that a solution of equation (5) is a function $v = v(t)$ whose graph is a curve in the tv-plane. The importance of Figure 1.1.3 is that each line segment is a tangent line to one of these solution curves. Thus, even though we have not found any solutions, and no graphs of solutions appear in the figure, we can nonetheless draw some qualitative conclusions about the behavior of solutions. For instance, if v is less than a certain critical value, then all the line segments have positive slopes, and the speed of the falling object increases as it falls. On the other hand, if v is greater than the critical value, then the line segments have negative slopes, and the falling object slows down as it falls. What is this critical value of v that separates objects whose speed is increasing from those whose speed is decreasing? Referring again to equation (5), we ask what value of v will cause dv/dt to be zero. The answer is $v = (5)(9.8) = 49$ m/s.

In fact, the constant function $v(t) = 49$ is a solution of equation (5). To verify this statement, substitute $v(t) = 49$ into equation (5) and observe that each side of the equation is zero. Because it does not change with time, the solution $v(t) = 49$ is called an **equilibrium solution**. It is the solution that corresponds to a perfect balance between gravity and drag. In Figure 1.1.3 we show the equilibrium solution $v(t) = 49$ superimposed on the direction field. From this figure we can draw another conclusion, namely, that all other solutions seem to be converging to the equilibrium solution as t increases. Thus, in this context, the equilibrium solution is often called the **terminal velocity**.

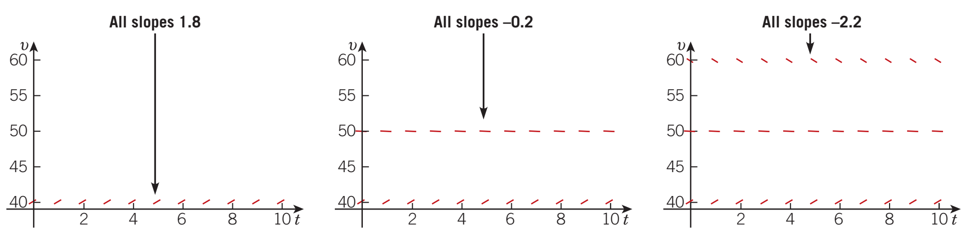

FIGURE 1.1.2 Assembling a direction field for equation (5): $dv/dt = 9.8 - v/5$. (a) when $v = 40$, $dv/dt = 1.8$, (b) when $v = 50$, $dv/dt = -0.2$, and (c) when $v = 60$, $dv/dt = -2.2$.

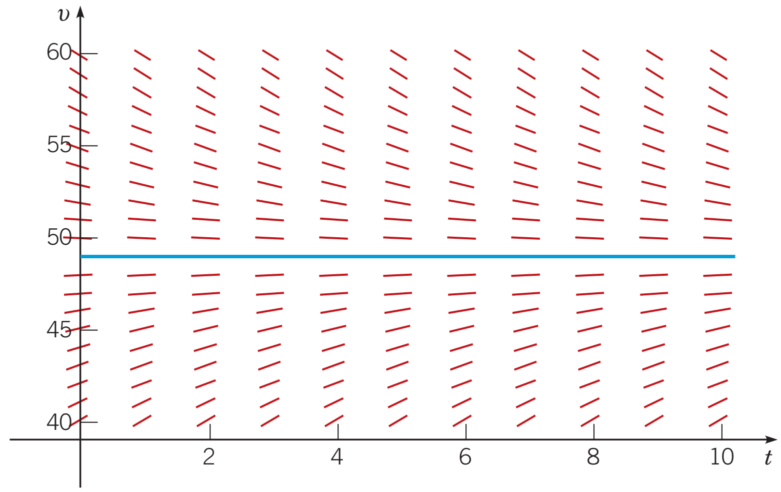

FIGURE 1.1.3 Direction field and equilibrium solution for equation (5): $dv/dt = 9.8 - v/5$.

The approach illustrated in Example 2 can be applied equally well to the more general differential equation (4), where the parameters m and γ are unspecified positive numbers. The results are essentially identical to those of Example 2. The equilibrium solution of equation (4) is the constant solution $v(t) = mg/\gamma$. Solutions below the equilibrium solution increase with time, and those above it decrease with time. As a result, we conclude that all solutions approach the equilibrium solution as t becomes large.

Direction Fields. Direction fields are valuable tools in studying the solutions of differential equations of the form

$$\frac{dy}{dt} = f(t, y), \tag{6}$$

where f is a given function of the two variables t and y, sometimes referred to as the **rate function**. A direction field for equations of the form (6) can be constructed by evaluating f at each point of a rectangular grid. At each point of the grid, a short line segment is drawn whose slope is the value of f at that point. Thus each line segment is tangent to the graph of the solution passing through that point. A direction field drawn on a fairly fine grid gives a good picture of the overall behavior of solutions of a differential equation. Usually a grid consisting of a few hundred points is sufficient. The construction of a direction field is often a useful first step in the investigation of a differential equation.

Two observations are worth particular mention. First, in constructing a direction field, we do not have to solve equation (6); we just have to evaluate the given function $f(t, y)$ many times. Thus direction fields can be readily constructed even for equations that may be quite difficult to solve. Second, repeated evaluation of a given function and drawing a direction field are tasks for which a computer or other computational or graphical aid are well suited. All the direction fields shown in this book, such as the one in Figures 1.1.2 and 1.1.3, were computer generated.

Field Mice and Owls. Now let us look at another, quite different example. Consider a population of field mice that inhabit a certain rural area. In the absence of predators we assume that the mouse population increases at a rate proportional to the current population. This assumption is not a well-established physical law (as Newton's law of motion is in Example 1), but it is a common initial hypothesis[1] in a study of population growth. If we denote time by t and the mouse population at time t by $p(t)$, then the assumption about population growth can be expressed by the equation

$$\frac{dp}{dt} = rp, \tag{7}$$

[1] A better model of population growth is discussed in Section 2.5.

where the proportionality factor r is called the **rate constant** or **growth rate**. To be specific, suppose that time is measured in months and that the rate constant r has the value 0.5/month. Then the two terms in equation (7) have the units of mice/month.

Now let us add to the problem by supposing that several owls live in the same neighborhood and that they kill 15 field mice per day. To incorporate this information into the model, we must add another term to the differential equation (7), so that it becomes

$$\frac{dp}{dt} = \frac{p}{2} - 450. \tag{8}$$

Observe that the predation term is -450 rather than -15 because time is measured in months, so the monthly predation rate is needed.

EXAMPLE 3

Investigate the solutions of differential equation (8) graphically.

Solution:

A direction field for equation (8) is shown in Figure 1.1.4. For sufficiently large values of p it can be seen from the figure, or directly from equation (8) itself, that dp/dt is positive, so that solutions increase. On the other hand, if p is small, then dp/dt is negative and solutions decrease. Again, the critical value of p that separates solutions that increase from those that decrease is the value of p for which dp/dt is zero. By setting dp/dt equal to zero in equation (8) and then solving for p, we find the equilibrium solution $p(t) = 900$, for which the growth term and the predation term in equation (8) are exactly balanced. The equilibrium solution is also shown in Figure 1.1.4.

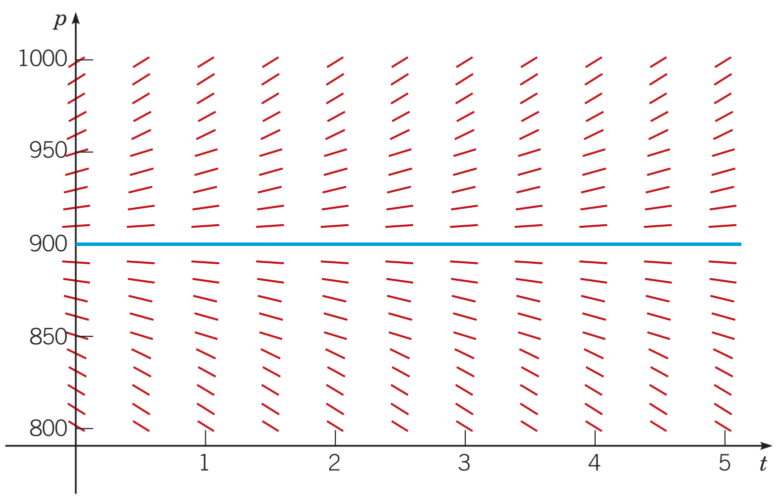

FIGURE 1.1.4 Direction field (red) and equilibrium solution (blue) for equation (8): $dp/dt = p/2 - 450$.

Comparing Examples 2 and 3, we note that in both cases the equilibrium solution separates increasing from decreasing solutions. In Example 2 other solutions converge to, or are attracted by, the equilibrium solution, so that after the object falls long enough, an observer will see it moving at very nearly the equilibrium velocity. On the other hand, in Example 3 other solutions diverge from, or are repelled by, the equilibrium solution. Solutions behave very differently depending on whether they start above or below the equilibrium solution. As time passes, an observer might see populations either much larger or much smaller than the equilibrium population, but the equilibrium solution itself will not, in practice, be observed. In both problems, however, the equilibrium solution is very important in understanding how solutions of the given differential equation behave.

A more general version of equation (8) is

$$\frac{dp}{dt} = rp - k, \tag{9}$$

where the growth rate r and the predation rate k are positive constants that are otherwise unspecified. Solutions of this more general equation are very similar to those of equation (8). The equilibrium solution of equation (9) is $p(t) = k/r$. Solutions above the equilibrium solution increase, while those below it decrease.

You should keep in mind that both of the models discussed in this section have their limitations. The model (5) of the falling object is valid only as long as the object is falling freely, without encountering any obstacles. If the velocity is large enough, the assumption that the frictional resistance is linearly proportional to the velocity has to be replaced with a nonlinear approximation (see Problem 21). The population model (8) eventually predicts negative numbers of mice (if $p < 900$) or enormously large numbers (if $p > 900$). Both of these predictions are unrealistic, so this model becomes unacceptable after a fairly short time interval.

Constructing Mathematical Models. In applying differential equations to any of the numerous fields in which they are useful, it is necessary first to formulate the appropriate differential equation that describes, or models, the problem being investigated. In this section we have looked at two examples of this modeling process, one drawn from physics and the other from ecology. In constructing future mathematical models yourself, you should recognize that each problem is different, and that successful modeling cannot be reduced to the observance of a set of prescribed rules. Indeed, constructing a satisfactory model is sometimes the most difficult part of the problem. Nevertheless, it may be helpful to list some steps that are often part of the process:

1. Identify the independent and dependent variables and assign letters to represent them. Often the independent variable is time.
2. Choose the units of measurement for each variable. In a sense the choice of units is arbitrary, but some choices may be much more convenient than others. For example, we chose to measure time in seconds for the falling-object problem and in months for the population problem.
3. Articulate the basic principle that underlies or governs the problem you are investigating. This may be a widely recognized physical law, such as Newton's law of motion, or it may be a more speculative assumption that may be based on your own experience or observations. In any case, this step is likely not to be a purely mathematical one, but will require you to be familiar with the field in which the problem originates.
4. Express the principle or law in step 3 in terms of the variables you chose in step 1. This may be easier said than done. It may require the introduction of physical constants or parameters (such as the drag coefficient in Example 1) and the determination of appropriate values for them. Or it may involve the use of auxiliary or intermediate variables that must then be related to the primary variables.
5. If the units agree, then your equation at least is dimensionally consistent, although it may have other shortcomings that this test does not reveal.
6. In the problems considered here, the result of step 4 is a single differential equation, which constitutes the desired mathematical model. Keep in mind, though, that in more complex problems the resulting mathematical model may be much more complicated, perhaps involving a system of several differential equations, for example.

Historical Background, Part I: Newton, Leibniz, and the Bernoullis. Without knowing something about differential equations and methods of solving them, it is difficult to appreciate the history of this important branch of mathematics. Further, the development of differential equations is intimately interwoven with the general development of mathematics and cannot be separated from it. Nevertheless, to provide some historical perspective, we indicate here some of the major trends in the history of the subject and identify the most prominent early contributors. The rest of the historical background in this section focuses on the earliest contributors from the seventeenth century. The story continues at the end of Section 1.2 with an overview of the contributions of Euler and other eighteenth-century (and early-nineteenth-century) mathematicians. More recent advances, including the use of computers and other technologies, are summarized at the end of Section 1.3. Additional historical information is contained in footnotes scattered throughout the book and in the references listed at the end of the chapter.

The subject of differential equations originated in the study of calculus by Isaac Newton (1643–1727) and Gottfried Wilhelm Leibniz (1646–1716) in the seventeenth century. Newton grew up in the English countryside, was educated at Trinity College, Cambridge, and became Lucasian Professor of Mathematics there in 1669. His epochal discoveries of calculus and of the fundamental laws of mechanics date to 1665. They were circulated privately among his friends, but Newton was extremely sensitive to criticism and did not begin to publish his results until 1687 with the appearance of his most famous book *Philosophiae Naturalis Principia Mathematica*. Although Newton did relatively little work in differential equations as such, his development of the calculus and elucidation of the basic principles of mechanics provided a basis for their applications in the eighteenth century, most notably by Euler (see Historical Background, Part II in Section 1.2). Newton identified three forms of first-order differential equations: $dy/dx = f(x)$, $dy/dx = f(y)$, and $dy/dx = f(x, y)$. For the latter equation he developed a method of solution using infinite series when $f(x, y)$ is a polynomial in x and y. Newton's active research in mathematics ended in the early 1690s, except for the solution of occasional "challenge problems" and the revision and publication of results obtained much earlier. He was appointed Warden of the British Mint in 1696 and resigned his professorship a few years later. He was knighted in 1705 and, upon his death in 1727, became the first scientist buried in Westminster Abbey.

Leibniz was born in Leipzig, Germany, and completed his doctorate in philosophy at the age of 20 at the University of Altdorf. Throughout his life he engaged in scholarly work in several different fields. He was mainly self-taught in mathematics, since his interest in this subject developed when he was in his twenties. Leibniz arrived at the fundamental results of calculus independently, although a little later than Newton, but was the first to publish them, in 1684. Leibniz was very conscious of the power of good mathematical notation and was responsible for the notation dy/dx for the derivative and for the integral sign. He discovered the method of separation of variables (Section 2.2) in 1691, the reduction of homogeneous equations to separable ones (Section 2.2, Problem 30) in 1691, and the procedure for solving first-order linear equations (Section 2.1) in 1694. He spent his life as ambassador and adviser to several German royal families, which permitted him to travel widely and to carry on an extensive correspondence with other mathematicians, especially the Bernoulli brothers. In the course of this correspondence many problems in differential equations were solved during the latter part of the seventeenth century.

The Bernoulli brothers, Jakob (1654–1705) and Johann (1667–1748), of Basel, Switzerland did much to develop methods of solving differential equations and to extend the range of their applications. Jakob became professor of mathematics at Basel in 1687, and Johann was appointed to the same position upon his brother's death in 1705. Both men were quarrelsome, jealous, and frequently embroiled in disputes, especially with each other. Nevertheless, both also made significant contributions to several areas of mathematics. With the aid of calculus, they solved a number of problems in mechanics by formulating them as differential equations. For example, Jakob Bernoulli solved the differential equation $y' = \left(a^3/(b^2y - a^3)\right)^{1/2}$ (see Problem 9 in Section 2.2) in 1690 and, in the same paper, first used the term "integral" in the modern sense. In 1694 Johann Bernoulli was able to solve the equation $dy/dx = y/(ax)$ (see Problem 10 in Section 2.2). One problem that both brothers solved, and that led to much friction between them, was the **brachistochrone problem** (see Problem 24 in Section 2.3). The brachistochrone problem was also solved by Leibniz, Newton, and the Marquis de l'Hôpital. It is said, perhaps apocryphally, that Newton learned of the problem late in the afternoon of a tiring day at the Mint and solved it that evening after dinner. He published the solution anonymously, but upon seeing it, Johann Bernoulli exclaimed, "Ah, I know the lion by his paw."

Daniel Bernoulli (1700–1782), son of Johann, migrated to St. Petersburg, Russia, as a young man to join the newly established St. Petersburg Academy, but returned to Basel in 1733 as professor of botany and, later, of physics. His interests were primarily in partial differential equations and their applications. For instance, it is his name that is associated with the Bernoulli equation in fluid mechanics. He was also the first to encounter the functions that a century later became known as Bessel functions (Section 5.7).

Problems

In each of Problems 1 through 4, draw a direction field for the given differential equation. Based on the direction field, determine the behavior of y as $t \to \infty$. If this behavior depends on the initial value of y at $t = 0$, describe the dependency.

G **1.** $y' = 3 - 2y$

G **2.** $y' = 2y - 3$

G **3.** $y' = -1 - 2y$

G **4.** $y' = 1 + 2y$

In each of Problems 5 and 6, write down a differential equation of the form $dy/dt = ay + b$ whose solutions have the required behavior as $t \to \infty$.

5. All solutions approach $y = 2/3$.

6. All other solutions diverge from $y = 2$.

In each of Problems 7 through 10, draw a direction field for the given differential equation. Based on the direction field, determine the behavior of y as $t \to \infty$. If this behavior depends on the initial value of y at $t = 0$, describe this dependency. Note that in these problems the equations are not of the form $y' = ay + b$, and the behavior of their solutions is somewhat more complicated than for the equations in the text.

G **7.** $y' = y(4 - y)$

G **8.** $y' = -y(5 - y)$

G **9.** $y' = y^2$

G **10.** $y' = y(y - 2)^2$

Consider the following list of differential equations, some of which produced the direction fields shown in Figures 1.1.5 through 1.1.10. In each of Problems 11 through 16, identify the differential equation that corresponds to the given direction field.

a. $y' = 2y - 1$
b. $y' = 2 + y$
c. $y' = y - 2$
d. $y' = y(y + 3)$
e. $y' = y(y - 3)$
f. $y' = 1 + 2y$
g. $y' = -2 - y$
h. $y' = y(3 - y)$
i. $y' = 1 - 2y$
j. $y' = 2 - y$

11. The direction field of Figure 1.1.5.

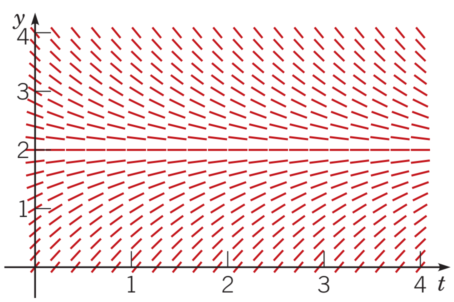

FIGURE 1.1.5 Problem 11.

12. The direction field of Figure 1.1.6.

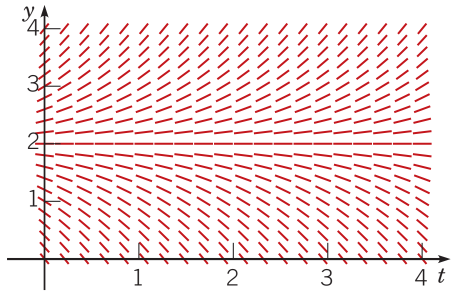

FIGURE 1.1.6 Problem 12.

13. The direction field of Figure 1.1.7.

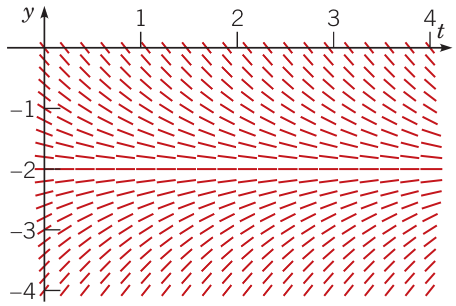

FIGURE 1.1.7 Problem 13.

14. The direction field of Figure 1.1.8.

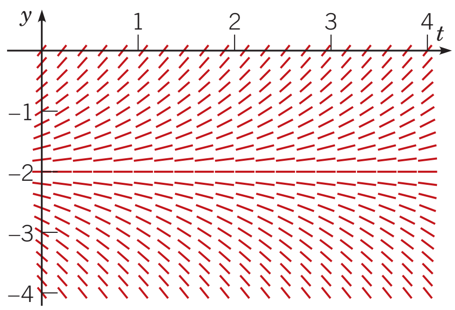

FIGURE 1.1.8 Problem 14.

15. The direction field of Figure 1.1.9.

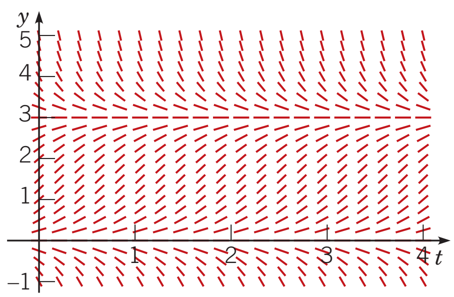

FIGURE 1.1.9 Problem 15.

16. The direction field of Figure 1.1.10.

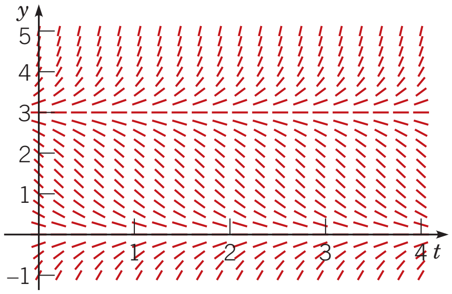

FIGURE 1.1.10 Problem 16.

17. A pond initially contains 1,000,000 gal of water and an unknown amount of an undesirable chemical. Water containing 0.01 grams of this chemical per gallon flows into the pond at a rate of 300 gal/h. The mixture flows out at the same rate, so the amount of water in the pond remains constant. Assume that the chemical is uniformly distributed throughout the pond.

a. Write a differential equation for the amount of chemical in the pond at any time.
b. How much of the chemical will be in the pond after a very long time? Does this limiting amount depend on the amount that was present initially?
c. Write a differential equation for the concentration of the chemical in the pond at time t. *Hint:* The concentration is $c = a/v = a(t)/10^6$.

18. A spherical raindrop evaporates at a rate proportional to its surface area. Write a differential equation for the volume of the raindrop as a function of time.

19. Newton's law of cooling states that the temperature of an object changes at a rate proportional to the difference between the temperature of the object itself and the temperature of its surroundings (the ambient air temperature in most cases). Suppose that the ambient temperature is 70°F and that the rate constant is 0.05 $(\text{min})^{-1}$. Write a differential equation for the temperature of the object at any time. Note that the differential equation is the same whether the temperature of the object is above or below the ambient temperature.

20. A certain drug is being administered intravenously to a hospital patient. Fluid containing 5 mg/cm^3 of the drug enters the patient's bloodstream at a rate of 100 cm^3/h. The drug is absorbed by body tissues or otherwise leaves the bloodstream at a rate proportional to the amount present, with a rate constant of 0.4/h.

a. Assuming that the drug is always uniformly distributed throughout the bloodstream, write a differential equation for the amount of the drug that is present in the bloodstream at any time.
b. How much of the drug is present in the bloodstream after a long time?

N 21. For small, slowly falling objects, the assumption made in the text that the drag force is proportional to the velocity is a good one. For larger, more rapidly falling objects, it is more accurate to assume that the drag force is proportional to the square of the velocity.[2]

a. Write a differential equation for the velocity of a falling object of mass m if the magnitude of the drag force is proportional to the square of the velocity and its direction is opposite to that of the velocity.
b. Determine the limiting velocity after a long time.
c. If $m = 10$ kg, find the drag coefficient so that the limiting velocity is 49 m/s.
N d. Using the data in part c, draw a direction field and compare it with Figure 1.1.3.

In each of Problems 22 through 25, draw a direction field for the given differential equation. Based on the direction field, determine the behavior of y as $t \to \infty$. If this behavior depends on the initial value of y at $t = 0$, describe this dependency. Note that the right-hand sides of these equations depend on t as well as y; therefore, their solutions can exhibit more complicated behavior than those in the text.

G 22. $y' = -2 + t - y$

G 23. $y' = e^{-t} + y$

G 24. $y' = 3\sin t + 1 + y$

G 25. $y' = -\dfrac{2t + y}{2y}$

[2]See Lyle N. Long and Howard Weiss, "The Velocity Dependence of Aerodynamic Drag: A Primer for Mathematicians," *American Mathematical Monthly 106* (1999), 2, pp. 127–135.

1.2 Solutions of Some Differential Equations

In the preceding section we derived the differential equations

$$m\frac{dv}{dt} = mg - \gamma v \tag{1}$$

and

$$\frac{dp}{dt} = rp - k. \tag{2}$$

Equation (1) models a falling object, and equation (2) models a population of field mice preyed on by owls. Both of these equations are of the general form

$$\frac{dy}{dt} = ay - b, \tag{3}$$

where a and b are given constants. We were able to draw some important qualitative conclusions about the behavior of solutions of equations (1) and (2) by considering the associated direction fields. To answer questions of a quantitative nature, however, we need to find the solutions themselves, and we now investigate how to do that.